Applied Optimization

Problem:- A rectangular garden with 100 ft fencing. Determine the maximum area & the corresponding dimensions.

Solution:- (1) Introduce all variables.

Let x & y be the sides of the rectangular

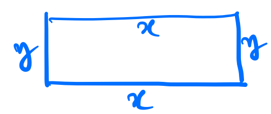
(2) Understand what you are trying to Maximize or Minimize

We want the area, say A, to maximize.

We know formula for area : $A = xy$.

(3) Use constraints to boil down to a function of only one variable.

Here fencing is 100 ft. So, sum of sides is 100ft



So, $x + y + x + y = 100$

$\Rightarrow 2(x+y) = 100$

$\Rightarrow x + y = 50$

$\Rightarrow y = 50 - x$

Then $A = xy = x(50 - x)$.

(4) Using Calculus techniques, find max/min of the target quantity.

$A = x(50-x) = 50x - x^2$ ———————— (*)

$A = A(x)$

Find critical pts by putting $A'(x) = 0$ or undefined.

Here, $A'(x) = 50 - 2x$, $A''(x) = -2 < 0$

So, $A'(x) = 0 \Rightarrow 50 - 2x = 0 \Rightarrow x = 25$

At critical pt, $A(x)$ attains max/min.

So, $A(25) = 50 \times 25 - 25 \times 25$ [by (*)].

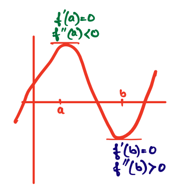
$= (50-25) \times 25$

$= 25 \times 25$

$= 625$ sqft. ⇝ Max area.

<u>Note</u>:- At max pt $x=a$, $f'(a) = 0$ & $f''(a) < 0$

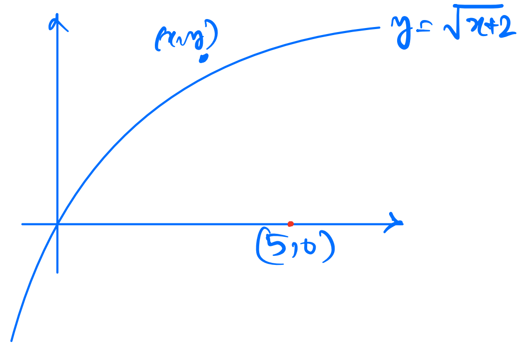
min pt $x=b$, $f'(b) = 0$ & $f''(b) > 0$



Dimensions: $x = 25$ ft, $y = 50 - 25 = 25$ ft.

Problem:- Find the closest point to $(5,0)$ on the curve $y=\sqrt{x+2}$, $x \geq -2$

Sol^n:- Choose a pt on $y=\sqrt{x+2}$. Say (x,y).



Distance between (x,y) & $(5,0)$ is

$$D=\sqrt{(x-5)^2+(y-0)^2}$$

$$=\sqrt{(x-5)^2+y^2}$$

Now, (x,y) is a point on $y=\sqrt{x+2}$, so

$$y=\sqrt{x+2}$$

Then $D=\sqrt{(x-5)^2+y^2}$ can be written in one variable x as:

$$D(x)=\sqrt{(x-5)^2+(\sqrt{x+2})^2}$$

$$=\sqrt{(x-5)^2+(x+2)}$$

$$=\sqrt{x^2-10x+25+x+2}$$

$$=\sqrt{x^2-9x+27}.$$

$$=(x^2-9x+27)^{1/2}$$

$$D'(x)=\frac{1}{2}(x^2-9x+27)^{-1/2}\cdot(2x-9)$$

$$=\frac{2x-9}{2\sqrt{x^2-9x+27}}$$

$D'(x)=0$ or $D'(x)$ undefined gives

$D'(x) = 0$

$\Downarrow$

$2x - 9 = 0$

$\Rightarrow x = 4.5$ critical pt

$D'(x) \rightsquigarrow$ undefined

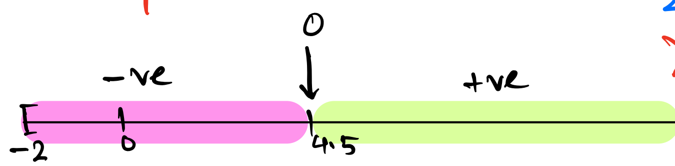
$\Downarrow$

$x^2 - 9x + 27 = 0$

$\Rightarrow x = \dfrac{9 \pm \sqrt{81 - 4 \cdot 1 \cdot 27}}{2 \cdot 1}$ $\quad \sqrt{-ve}$

✗

Note: $D'(x)$



with increasing x, $D'(x)$ is increasing

$\Rightarrow D''(x) > 0$

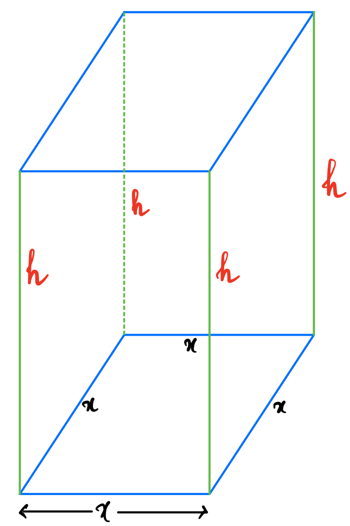
$\Rightarrow D(x)$ has a minima.

So, $x = 4.5 \Rightarrow y = \sqrt{x+2} = \sqrt{6.5}$

Hence, $(4.5, \sqrt{6.5})$ is the closest pt on $y = \sqrt{x+2}$ from $(5, 0)$

Problem: A box with a square base & an open top is to be built with a volume of 40 m^3. The materials for the bottom of the box costs \$10/m^2 & the materials on the sides costs \$8/m^2. Find the dimensions of the box for which the cost will be minimum.

Sol^n: **Note:** We need to minimize the Cost, so we must get an expression for Cost.



Area of the base $= (x \times x)\ m^2$
$= x^2\ m^2$

So, cost for the base $= (\$10 \times x^2)$

Area of one side $= (x \times h)\ m^2$
So, total area of sides $= (4xh)\ m^2$

Then total cost for sides $= (\$8 \times 4xh)$

Hence cost $C = (10x^2) + (32xh)$ dollars. ——— ⊛

Note: The volume of the box is fixed.

Our constraint is: Volume $= 40 m^3$

ie, $\underbrace{x \times x}_{\text{base}} \times \underbrace{h}_{\text{height}} = 40$

$\Rightarrow x^2 h = 40 \Rightarrow h = \frac{40}{x^2}$.

Then by ⊛ & the above constraint, we can reduce cost function to a function of single variable x, rather than two variables x & h.

So, $C(x) = 10x^2 + 32x\left(\frac{40}{x^2}\right)$

$$= 10x^2 + \frac{32\times 40}{x} = 10x^2 + (32\times 40)x^{-1}$$

To get maximum or minimum values of $C(x)$, we need the Critical pts.

$$C'(x) = 0 \Rightarrow \left[10x^2 + (32\times 40)x^{-1}\right]' = 0$$

$$\Rightarrow 10(x^2)' + (32\times 40)(x^{-1})' = 0$$

$$\Rightarrow 10(2x) + (32\times 40)(-x^{-2}) = 0$$

$$\Rightarrow 20x - \frac{32\times 40}{x^2} = 0$$

$$\Rightarrow 20x = \frac{32\times 40}{x^2}$$

$$\Rightarrow x^3 = \frac{32\times 40}{20} = 32\times 2 = 64 = 4^3$$

$$\Rightarrow \boxed{x = 4}$$

Then, $h = \frac{40}{x^2} = \frac{40}{16} = 2.5$

So the dimensions corresponding to the minimum cost is square base of 4m sides & height of 2.5m.

Hence the Cost is \$ $10(4)^2 + (32\times 40)(4^{-1})$

$$= 10(16) + \frac{32\times 40}{4}$$

$$= 160 + 32\times 10$$

$$= \$480.$$